Solutions of DC OPF are Never AC Feasible

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***Abstract*—In this paper, we analyze the relationship between generation dispatch solutions produced by the DC optimal power flow (DC OPF) problem by the AC optimal power flow (AC OPF) problem. While there has been much previous work in analyzing the approximation error of the DC assumption, the AC feasibility has not been fully explored, although difficulty achieving AC feasibility is known in practice. Here, under some light assumptions, we show that no solution to the DC OPF problem will satisfy the AC power flow constraints. Then, it is demonstrated that even with generation adjustments in DC OPF to account for losses, DC OPF solutions are still not AC feasible. Lastly, the computational benefits of DC OPF are analyzed in comparison with AC OPF.**

I. Introduction

Up to billions of dollars annually are lost to the suboptimal operation of electric power grids [1], and estimates of global carbon dioxide equivalents due to transmission and distribution losses can total one billion metric tons per year [2]. Pursuing optimal operation of grid assets and harnessing accurate and physically realizable models is essential to lowering both the financial and environmental impact of grid operations. The DC optimal power flow (DC OPF) approximation of the AC optimal power flow (AC OPF) problem is widely used for current power system operations due to its convexity and computational benefits. However, it is well-known that the DC OPF can provide, in some cases, a poor approximation of actual AC power flows and locational marginal prices (LMPs), potentially resulting in physically unrealizable system states, higher system losses, and lowered economic efficiency.

There has been much previous work in analyzing the approximation error between DC OPF and AC OPF solutions [3], [4], [5], [6]. For example, DC OPF is often used to calculate market prices, and previous works have looked at the difference in LMPs [7], [8] resulting from DC OPF and AC OPF. Outside of analyzing the differences in objective function value and approximation error between DC and AC OPF, it is important to consider the physical feasibility of the DC OPF solution. Namely, under which circumstances does the solution to a DC OPF problem produce a physically realizable solution (i.e., one that satisfies the AC power flow equations)? The difficulty of obtaining an AC feasible solution from DC OPF is well known; for example, in [9] the authors state "a solution of DC OPF may not be feasible," and in [10] the authors state that the DC OPF solution is "typically AC infeasible." In this note, we show that these statements can actually be made much stronger, because the solution to the standard DC OPF problem is actually *always* AC infeasible. Considering the origin of these differences may help us gain insight into how to address the AC feasibility of DC OPF.

We show that under some light assumptions, the set of points within the feasible region of the DC OPF problem and the set of points within the feasible region of the AC OPF problem have an empty intersection. While this has been observed in practice, we demonstrate mathematically why the DC OPF is never AC feasible. In addition, we show that even if the generation under DC equals that from AC (e.g. by using fictitious load models, for example), the remainder of the DC OPF solution is still not AC feasible. This has implications for future power system operation, considering many system operators still use DC OPF and are thus always required to go through an additional iterative procedure to ensure AC feasibility [11]. Towards the aforementioned issues, this paper offers two main contributions:

- We present what we believe is the first formal analysis demonstrating there is no overlap between the DC OPF and AC OPF feasible regions. The benefit of performing a mathematical analysis here rather than further relying on heuristics and physical observations is that it offers us insights into origins of the feasibility gap.
- We further show that even if the total generation in AC and DC OPF is made equal by inclusion of fictitious nodal demand or another technique, the solution obtained by DC OPF does not satisfy the AC power flow equations.

Generally, the solution of a nonlinear system of equations (AC power flow) will not be equal to the solution of a linear system of equations (DC power flow). However, this note emphasizes that in this particular case, they are *never* equal; indeed, DC power flow is not a real "linearization" of the AC power flow equations, despite often being titled as such; it is instead an "approximation." By analyzing both the AC feasibility of DC OPF and the computational gains provided by DC OPF, we can make a more critical assessment regarding the applicability of DC OPF to various operational scenarios.

II. Optimal Dispatch and Power Flow

To provide the necessary background to discuss the feasibility of the DC OPF solutions in the AC OPF problem, we first briefly summarize these problems which seek to optimize generation dispatch in transmission networks. First, define coefficients a_j, b_j, and c_j as the operational costs associated with generator j. Let set $\mathcal{G}$ be the set of all generators in the network, $\mathcal{N}$ be the set of all nodes (buses) in the network, $\mathcal{L}$ be the set of all lines (branches) in the network, and $\mathcal{G}_i$ be the set of generators connected to bus i. Define $p_{l,j}$ ($q_{l,j}$) as the total active (reactive) power consumption at node j, $p_{g,j}$ ($q_{g,j}$) is the active (reactive) power output of generator j, and $\underline{p}_{g,j}$ ($\underline{q}_{g,j}$) and $\overline{p}_{g,j}$ ($\overline{q}_{g,j}$) are lower and upper limits on

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active (reactive) power generation, respectively. The complex voltage at node i has magnitude $|v_i|$ and phase angle θ_i, and the difference in phase angle between neighboring buses i and m is written as θ_{im}. Values G_{im} and B_{im} are real and reactive parts of entry (i, m) in the admittance matrix, respectively.

A. AC Optimal Power Flow

The AC Optimal Power Flow (AC OPF) model is typically considered the ground truth for estimating physical power flows throughout the network, as it includes line losses, network parameters, and reactive power. It can be written with the AC power flow equations in polar form as follows:

$$\min_{\mathbf{v},\mathbf{p}_g} \sum_{j\in\mathcal{G}} a_j p_{g,j}^2 + b_j p_{g,j} + c_j \tag{1a}$$

s.t :

$$|v_i| \sum_{m\in\mathcal{N}} |v_m|(G_{im}\cos(\theta_{im}) + B_{im}\sin(\theta_{im})) = p_{l,i} - \sum_{k\in\mathcal{G}_i} p_{g,k}, \quad \forall i \in \mathcal{N} \tag{1b}$$

$$|v_i| \sum_{m\in\mathcal{N}} |v_m|(G_{im}\sin(\theta_{im}) - B_{im}\cos(\theta_{im})) = q_{l,i} - \sum_{k\in\mathcal{G}_i} q_{g,k}, \quad \forall i \in \mathcal{N} \tag{1c}$$

$$\underline{p}_{g,j} \le p_{g,j} \le \overline{p}_{g,j}, \quad \forall j \in \mathcal{G} \tag{1d}$$

$$\underline{q}_{g,j} \le q_{g,j} \le \overline{q}_{g,j}, \quad \forall j \in \mathcal{G} \tag{1e}$$

$$\underline{|v|} \le |v_i| \le \overline{|v|}, \quad \forall i \in \mathcal{N}. \tag{1f}$$

where $\mathbf{p}_g$ is a vector comprising the active power generation $p_{g,j}$ at each generator $j \in \mathcal{G}$ and $\mathbf{v}$ is $2n$-dimensional vector comprising the unknown voltage magnitudes and angles. It is clear from (1b) and (1c) that the overall optimization problem is nonconvex. Note that here we omit line flow constraints, but these can be modeled as constraints on apparent power flows, line currents, or voltage angle differences [12].

B. DC Optimal Power Flow

The constraints within the DC Optimal Power Flow problem (DC OPF) are linear approximations of the actual nonlinear AC power flows. The DC approximation is derived from multiple physical assumptions and observations. First, in transmission networks, the line resistance R_{im} is typically significantly less than the line reactance X_{im}; thus, B_{im} can be approximated to $-\frac{1}{X_{im}}$. Second, the phase angle difference between any two buses is typically small and usually does not exceed $30°$. From this, we can use the small angle approximation to approximate $\sin(\theta_{im}) \approx \theta_{im}$ and $\cos(\theta_{im}) \approx 1$. Third, transmission level voltage magnitudes are typically very close to 1.0 p.u. during normal operation.

Lastly, from these initial assumptions, and confirmed by the fact that reactive power is a localized phenomenon that cannot travel long distances, we see that the magnitude of the reactive power flow on the lines (denote this as Q_{im} for line im) is significantly less than the magnitude of the active power flow on the lines (denote this as P_{im} for line im). Using these assumptions when studying the AC power flow equations, equations (1b) and (1c) simplify and we are left with the following DC OPF problem. Note that in some formulations, the objective can also be linear instead of quadratic.

$$\min_{\mathbf{p}_g} \sum_{j\in\mathcal{G}} a_j p_{g,j}^2 + b_j p_{g,j} + c_j \tag{2a}$$

$$\text{s.t}: \; p_{l,i} - \sum_{k\in\mathcal{G}_i} p_{g,k} = \sum_{m\in\mathcal{N}} B_{im}\theta_{im}, \quad \forall i \in \mathcal{N} \tag{2b}$$

$$-F_{im} \le B_{im}\theta_{im} \le F_{im}, \quad \forall im \in \mathcal{L} \tag{2c}$$

where F_{im} represents the limit on the magnitude of the line flows on line im. Note that physically, transmission line flows are limited by the amount of current that can safely flow through the line. We can write the current flow limit on line im, $|I_{im}|$, in terms of the complex power S_{im}, real power P_{im}, and reactive power Q_{im} flowing from bus i to bus m and the complex voltage at bus i:

$$|I_{im}| = \Big|\Big(\frac{S_{im}}{v_i}\Big)^*\Big| = \Big(\frac{\sqrt{(P_{im}^2 + Q_{im}^2)}}{|v_i|}\Big),$$

and by using the DC approximations stated above, namely that $P_{im} >> Q_{im}$ and $|v_i| = 1.0$ p.u., we can write the current flow limit in terms of power

$$|I_{im}| \approx \sqrt{P_{im}^2} \approx F_{im},$$

which justifies (2c). More information on the derivation of the DC approximation can be found in [13].

III. AC Feasibility

Towards determining the AC feasibility of DC OPF, define the set of feasible $\mathbf{p}_g$ satisfying (2) as $\mathcal{Y}_{\mathcal{DC}} \in \mathbb{R}^{|\mathcal{G}|}$, and the set of feasible $\mathbf{p}_g$ satisfying (1) as $\mathcal{Y}_{\mathcal{AC}} \in \mathbb{R}^{|\mathcal{G}|}$. Note that for the first part of the feasibility analysis (without loss adjustments), it is sufficient to consider generation only. Next, we state the assumptions used in the analysis and results in this paper, which are generally reasonable assumptions made in most AC OPF analyses.

Assumption 1: Assume that the loads in the network are modeled as constant (P, Q) loads [14].

Assumption 2: Assume that power is flowing on at least one line in the network; i.e., for all feasible solutions $\mathbf{x} \in \mathcal{Y}_{\mathcal{AC}}$, $\sum_{j\in\mathcal{G}} p_{g,j} > \sum_{i=1}^{\mathcal{N}} p_{l,i}$ due to line losses.

Assumption 3: Assume the syste admittance matrix is symmetric.

Assumption 4: Assume line resistances and reactances are positive.

From these, we show that $\mathcal{Y}_{\mathcal{DC}} \cap \mathcal{Y}_{\mathcal{AC}} = \emptyset$. Note that because DC OPF neglects reactive power flows, it is not as meaningful to analyze AC feasibility by using reactive power balance equations; instead, active power balance equations are used.

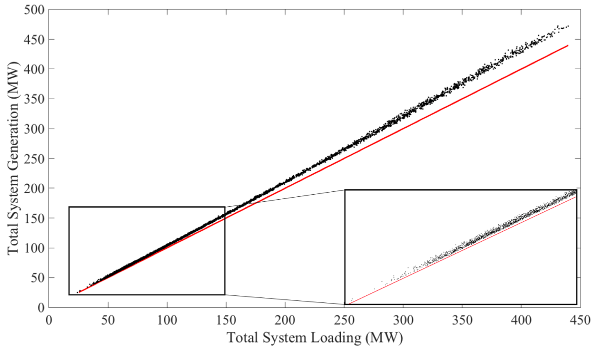

Fig. 1. Total generation active power output from DC OPF (red line) and AC OPF (black dots) for varying levels of total system loading. Due to tranmission losses, AC OPF always produces a solution where $\sum_{j\in\mathcal{G}} p_{g,j} > \sum_{i\in\mathcal{N}} p_{l,i}$ whereas DC OPF always produces a solution where $\sum_{j\in\mathcal{G}} p_{g,j} = \sum_{i\in\mathcal{N}} p_{l,i}$.

A. AC Feasibility of DC OPF

Consider the DC OPF problem as outlined in (2). The difference between economic dispatch (ED) and DC OPF is the consideration of line flows. However, notice that due to the lack of line losses in DC OPF, there exists a similarity between these two problems. In particular, note that the constraint $\sum_{j\in\mathcal{G}} p_{g,j} = \sum_{i\in\mathcal{N}} p_{l,i}$ holds for DC OPF. To illustrate this, consider constraint (2b). Summing the left-hand side and right-hand side over all $i \in \mathcal{N}$ yields

$$\sum_{i\in\mathcal{N}} p_{l,i} - \sum_{j\in\mathcal{G}} p_{g,j} = \sum_{i\in\mathcal{N}} \sum_{m\in\mathcal{N}} B_{im}\theta_{im}. \quad (3)$$

Since $B_{im} = B_{mi}$ and $\theta_{im} = -\theta_{mi}$, we are left with

$$\sum_{i\in\mathcal{N}} p_{l,i} - \sum_{j\in\mathcal{G}} p_{g,j} = 0. \quad (4)$$

Because of Assumption 2 and the presence of line losses, $\sum_{j\in\mathcal{G}} p_{g,j}$ will always be strictly greater than $\sum_{i\in\mathcal{N}} p_{l,i}$ in AC OPF. Losses have previously been identified as one of the main sources of DC approximation errors [5]; they are also the main cause of infeasibility, as shown here. Thus, $\mathcal{Y}_{\mathcal{DC}} \cap \mathcal{Y}_{\mathcal{AC}} = \emptyset$. Note that the generation values alone are enough to show this result, and we make no statements about the differences between voltage phase angles in the AC and DC cases.

To provide a simple illustration of the overall generation in each case, Figure 1 shows 5000 feasible runs of both DC and AC OPF for varying levels of random system loading in the IEEE 14-bus system using MATPOWER [15]. The red line in the figure shows the total system generation resulting from DC OPF, and the black dots show the total generation resulting from AC OPF. Multiple AC OPF solutions can correspond to one value of total system loading due to the inclusion of losses and distribution of the loading. Notice that this gap increases as system loading and losses increase.

Physically, no dispatch solution is "AC infeasible." Power flows will always adhere to physics, and if not enough generation is dispatched to meet the load, voltage issues or outages may occur. DC OPF solutions are often used in real systems to calculate marginal prices and often in academia to demonstrate algorithmic capabilities, limitations, or to perform research studies. Here, we attempt to further analyze sources of AC infeasibility for DC OPF.

IV. Feasibility of DC OPF with loss adjustments

Although often not implemented in research leveraging DC OPF formulations, in reality, of course, losses are not ignored. Generators will receive and implement the setpoints determined by DC OPF, and the slack bus will account for the mismatch between the DC OPF-determined generation setpoints and the actual network demand, for example. This will result in the solutions in Fig. 1 overlapping - total generation in the DC OPF will then equal total generation in the AC OPF. Losses can also be included by adding "fictitious nodal demand" by modifying the load at each bus in order to result in a solution closer to AC OPF [16], [17]. Now, assuming there is a situation where the generation dispatch solution provided by DC OPF is equal to that provided by AC OPF, we can analyze if angles within the DC OPF solution will be AC feasible, assuming the structure of the DC power flow equations does not change.

A. With small angle approximation

A DC OPF solution comprised of voltage angles and active power dispatch levels is considered "AC feasible" if it satisfies the AC power flow equations. We will show with a proof by contradiction that even with the slack bus accounting for system losses or artificially increasing demand, DC OPF results in an AC infeasible point. To that end, consider the active power balance AC power flow equation at some bus i evaluated at a DC OPF solution, where DC OPF necessitates $|v_i| = 1.0$ for all voltage magnitudes in the network.

$$\sum_{m\in\mathcal{N}} (G_{im}\cos(\theta_{im}) + B_{im}\sin(\theta_{im})) = p_{l,i} - \sum_{k\in\mathcal{G}_i} p_{g,k}. \quad (5)$$

Note that the right hand side, by virtue of the DC power flow equations being satisfied, can be rewritten as:

$$\sum_{m\in\mathcal{N}} (G_{im}\cos(\theta_{im}) + B_{im}\sin(\theta_{im})) = \sum_{m\in\mathcal{N}} B_{im}\theta_{im}. \quad (6)$$

This equality should hold if the DC solution (namely, $|V_i| = |V_m| = 1.0$ and θ_{im}, $\forall m \in \mathcal{N}$) are AC feasible. In this case, consider that the small angle approximation holds, and thus

$$\sum_{m\in\mathcal{N}} (G_{im} + B_{im}\theta_{im}) = \sum_{m\in\mathcal{N}} B_{im}\theta_{im}. \quad (7)$$

Since $G_{im} > 0$, this cannot hold and the solution from DC OPF is not AC feasible.

B. Relevance of voltage angles

It is worth noting that in practice, DC OPF (with or without a loss adjustment procedure) may be used to fix active power generation setpoints and determine LMPs. If this is the case, why does it matter to assess the AC feasibility of the voltage angles as well? With the loss adjustment, if the p_{g_j} variables are fixed to the values obtained by DC OPF and a reduced-dimension AC OPF is solved to obtain voltages, a feasible solution for voltages *may* exist (the same is not true for standard DC OPF). However, the voltage angles in DC OPF are a key part of the solution - they tell us estimates of line flows, network congestion, and provide indications of system security. The fact that these variables are providing indications of a system state that is not physically realizable is still an important consideration that grid operators may want to consider.

V. DC OPF Computational Benefits

The assumptions made to reduce the AC power flow equations to a linear system of equations (e.g. no line losses, no reactive power, fixed voltage magnitudes, and small phase angle differences) result in computational benefits that have allowed operators to solve OPF on real-time timescales (e.g. every 5-15 minutes). Here, we evaluate various networks under a variety of loading scenarios, and note the time to solve both the AC OPF and DC OPF problems (here, using MATPOWER). Examining pros and cons of DC OPF can allow grid operators to determine the tradeoffs they are willing to make between feasibility, accuracy, and time.

In this example, 100 feasible loading scenarios for each considered network were simulated on a 2017 MacBook Pro. The results and average (denoted by the diamond symbols) are show in Fig. 2 for 300, 1,354, 3,012, and 9,241 bus networks. Note the logarithmic scale on the x-axis. The benefit of DC OPF is clear - the computational time, especially for the larger networks, is much lower. However, even on a 9,241 bus network, AC OPF takes less than three minutes to solve in the worst case encountered. Three minutes is currently less time than the timescale in which OPF is typically solved in current systems. Note that an extensive analysis comparing solver performance on greater computing power for a small number of loading scenarios per network can be found in [18]. It is worth mentioning that the speedups from DC OPF are more noticeable when used in applications such as security-constrained OPF or OPF formulations that have integer variables (unit commitment, transformers, switchable shunts, etc.) although these applications are not simulated here.

VI. Conclusions

In this note, we mathematically demonstrated how a solution to the DC OPF, under some light assumptions, will never be able to satisfy the AC power flow equations. While this has been observed in practice, and the employment of iterative "AC feasibility" techniques are currently being used by system operators to modify the original DC OPF problem, it had not yet been shown that a solution for active power generation and voltages satisfying both DC and AC OPF does not exist.

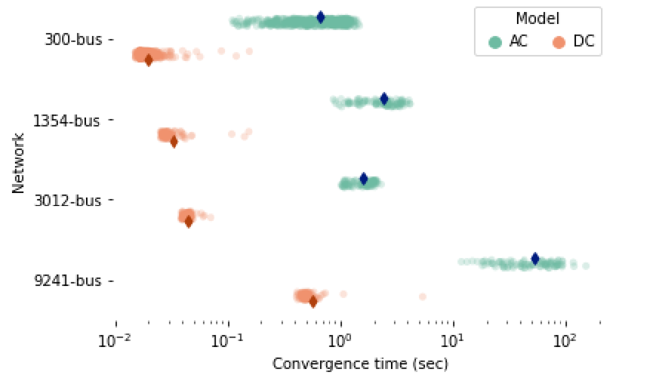

Fig. 2. Time in seconds to solve AC and DC OPF for 100 different loading scenarios and 4 different networks. Even for a 9,241 bus network, AC OPF can be solved in a couple of minutes. Diamonds denote the average across the scenarios.

Much previous work has been focused on quantifying and measuring the DC approximation error rather than analyzing AC feasibility. Additionally, with increases in computing power, AC OPF becomes a more and more tractable problem, and the benefits of using DC OPF for fast computation times is less motivating than it once was. However, linear models such as the DC power flow equations show great benefit under other optimization settings, such as stochastic OPF, security-constrained OPF, or other mixed-integer problems. Increasing renewable integration and uncertainty may also necessitate computationally efficient models.

Future work will address the assessment of power system security, economic efficiency, and greenhouse gas emissions resulting from the use of DC OPF versus AC OPF. Voltage issues in particular stemming from the use of DC OPF have also been observed in practice but the extent of this impact is an important future direction of this work.

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